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Dirac Mechanics and Landau Two-Fluid Model in ^{4}He II

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This paper is devoted to the development of the Dirac formalism for singular systems when applied to the Landau two-fluid model in superfluid helium. Notably, the Hamiltonian density is weakly zero (in the sense of Dirac). We obtain the physical and gauge variables, and show that all the constraints are of first class, and hence, that the Dirac bracket coincides with the Poisson bracket. We leave the quantization of this system for a later work.

1. INTRODUCTION

In this work we study some features that the Landau two-fluid model (1941) presents when is seen from the point of view of the Dirac mechanics (Dirac, 1950, 1951, 1958, 1964, 1966; Bergmann and Goldberg, 1955; Sudarshan and Mukunda, 1974).

The Lagrangian formulation of the equations of two-fluid hydrodynamics is given by Khalatnikov (1952). Pokrovsky and Khalatnikov (1976) establish those equations by using the usual Hamiltonian dynamics; but in their work they do not take into account the singularity of the Lagrangian density, i.e., the existence of constraints in the phase space.

We show that the Lagrangian density proposed by Khalatnikov (1952) is singular, so that to find the time evolution of the physical variables we must apply the Dirac formalism.

The Dirac mechanics is used by Chela-Flores et al. (to be published) to describe superfluid helium, but that description corresponds to the gauge theory given by Chela–Flores (1975, 1976) and is essentially different from the one we study here.

In Sections 2 and 3 we review some basic concepts of the Dirac mechanics and the Landau two-fluid model, respectively; in Section 4 we

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apply the Dirac formalism to the Landau–Khalatnikov model, and in Section 5 we state our conclusions.

2. SUMMARY OF THE DIRAC MECHANICS

We give the most important results of the Dirac mechanics of interest in our work.

Let $q=(q_1,q_2,\ldots,q_N)$ and $p=(p_1,p_2,\ldots,p_N)$ denote collectively the generalized coordinates and momenta of a system of particles. A Lagrangian is singular if and only if

$$\det\left\|\frac{\partial^2 L}{\partial \dot{q}_n \partial \dot{q}_{n'}}\right\|=0 \tag{2.1}$$

where n, n' take all values from 1 to N.

The primary constraints[1] $f^1_{a_1}(q,p)\approx 0$ $(a_1=1,\ldots,A)$ are those that arise from the definition

$$p_n=\frac{\partial L}{\partial \dot{q}_n} \tag{2.2}$$

and the secondary constraints $f^{k'}_{a_k'}(q,p)\approx 0$ $(k'\geqslant 2)$ are these that arise from the self-consistency equations

$$\dot{f}^k_{a_k}\approx\frac{\partial f^k_{a_k}}{\partial t}+\frac{\partial f^k_{a_k}}{\partial q_n}\dot{q}_n+\frac{\partial f^k_{a_k}}{\partial p_n}\dot{p}_n\approx 0 \qquad (k=1,\ldots,M) \tag{2.3}$$

where

$$\dot{q}_n\approx\frac{\partial H}{\partial p_n}+\sum_{a_1=1}^{A}u_{a_1}\frac{\partial f^1_{a_1}}{\partial p_n} \tag{2.4}$$

$$\dot{p}_n\approx-\frac{\partial H}{\partial q_n}-\sum_{a_1=1}^{A}u_{a_1}\frac{\partial f^1_{a_1}}{\partial q_n} \tag{2.5}$$

The u_{a_1} are noncanonical variables and can be determined from the self-consistency equations.

A variable $g(q,p)$ is *first class* if

$$\left\{g,f^k_{a_k}\right\}\approx 0 \qquad \forall a_k \tag{2.6}$$

[1] "$\approx$" means "weak equality." This terminology was introduced by Dirac to remind us that we must not use these constraints before working out a Poisson bracket.

i.e., if it has Poisson bracket weakly zero with all the constraints (primary and secondary). The variable $g(q,p)$ is *second class* in the contrary case.

Theorem. A quantity $g(q,p)$ is a physical variable, Dirac shows, if and only if

$$\{g, f_a\} \approx 0, \qquad a = 1, \ldots, R \tag{2.7}$$

with all the primary and first class constraints. In the contrary case, $g(q,p)$ is nonphysical.

Finally, the Dirac bracket is defined by

$$\{g, h\}^* = \{g, h\} - \Sigma\{g, f_i\} c^{ij} \{f_j, h\} \tag{2.8}$$

the sum being extended over the second class constraints of one irreducible descomposition of them, and C is the inverse matrix of the Poisson brackets $\{f_i, f_j\}$.

3. LANDAU TWO-FLUID MODEL

According to Landau theory, helium is composed of two liquids which can move independently: one liquid moves with velocity $\mathbf{v}_s$, which is responsible for superfluid properties, and another moves with velocity $\mathbf{v}_n$, which is responsible for the viscosity.

The Lagrangian density proposed by Khalatnikov is

$$\mathcal{L}(\rho, S, \mathbf{j}) = -\tfrac{1}{2}\rho v_s^2 + \mathbf{j}\cdot\mathbf{v}_s - \epsilon(\rho, S, \mathbf{v}_n - \mathbf{v}_s)$$

$$+ \alpha(\dot{\rho} + \nabla\cdot\mathbf{j}) + \beta\left[\dot{S} + \nabla\cdot(S\mathbf{v}_n)\right] + \nu\left[\dot{F} - \nabla\cdot(F\mathbf{v}_n)\right] \tag{3.1}$$

where

$$\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s \tag{3.2}$$

and ρ_n and ρ_s are the densities of the two liquids; S is the density of entropy; ϵ is the energy density, and $\mathbf{j}$ is the momentum density of the liquid.

We observe that in (3.1) appears the equations of motion as restrictions, with α, β, and ν as Lagrangian multipliers. However, it is unnecesary to obtain the equations of motion via Lagrange because they do not affect the variational principle[2]; hence, in order to simplify the calculus, we use

$$\mathcal{L}(\rho, S, \mathbf{p}) = \tfrac{1}{2}\rho v_s^2 + \mathbf{p}\cdot\mathbf{v}_s - \epsilon(\rho, S, \mathbf{p}) \tag{3.3}$$

with $\mathbf{p} = \rho_n(\mathbf{v}_n - \mathbf{v}_s) = S\nabla\beta + F\nabla\nu$.

An identical Lagrangian density is obtained by Lhuillier *et al.* (1975). We emphasize, however, that the procedure they employ is not valid in general. In effect, Kálnay and Ruggeri (1973) show that under certain conditions the quantization procedure is altered (and therefore the physical properties) when one adds a total time derivative to the Lagrangian of a constrained classical model.

Finally, Pokrovsky and Khalatnikov (1976) obtained the energy $\mathbf{H}$ of the fluid in the stationary coordinate frame:

$$\mathbf{H} = \int\left[\frac{1}{2}\rho v_s^2 + \mathbf{p}\cdot\mathbf{v}_s + \epsilon(\rho, S, \mathbf{p})\right]dv \tag{3.4}$$

and the three pairs of conjugate variables are

$$(\alpha, \rho) \qquad (\beta, S) \qquad (\nu, F) \tag{3.5}$$

4. APPLICATION OF THE DIRAC FORMALISM

Let the Lagrangian density be

$$\mathcal{L}(\rho, S, \mathbf{p}) = \tfrac{1}{2}\rho v_s^2 + \mathbf{p}\cdot\mathbf{v}_s - \epsilon(\rho, S, \mathbf{p}) \tag{4.1}$$

We note that $\mathcal{L}$ is singular, because in (4.1) the velocities do not appear explicitly.

[2]We recall that in the regular and the singular case, the variation of the action integral must be zero:

$$0 = \delta S = \delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i\right)dt = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)dt = 0$$

As we can observe, in order to obtain the motion equations, we only need the independence between the variations of q_i.

Constraints in the Phase Space. From the definition of the generalized momenta, we obtain that

$$\rho=\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}=0$$

$$S=\frac{\partial \mathcal{L}}{\partial \dot{\beta}}=0 \qquad (4.2)$$

$$F=\frac{\partial \mathcal{L}}{\partial \dot{\nu}}=0$$

thus, the system presents three primary constraints

$$f_1^1 \approx \rho \approx 0, \qquad f_2^1 \approx S \approx 0, \qquad f_3^1 \approx F \approx 0 \qquad (4.3)$$

Then, the velocities $\dot{\alpha}$, $\dot{\beta}$, and $\dot{\nu}$ cannot be expressed in terms of the generalized coordinates and momenta; this is a typical feature of singular systems. The pre-Hamiltonian h has the form

$$h(\alpha,\beta,\nu,\dot{\alpha},\dot{\beta},\dot{\nu},\rho,S,F,\nabla\alpha,\nabla\beta,\nabla\nu)=\rho\dot{\alpha}+S\dot{\beta}+F\dot{\nu}$$
$$-\tfrac{1}{2}\rho v_s^2-\mathbf{p}\cdot\mathbf{v}_s+\epsilon(\rho,S,\mathbf{p}) \qquad (4.4)$$

Thus, the usual method cannot be applied. However, using the constraints, we obtain the Hamiltonian density

$$\mathcal{H}(\alpha,\beta,\nu,\rho,S,F,\nabla\alpha,\nabla\beta,\nabla\nu)\approx 0 \qquad (4.5)$$

On the other hand, the equations of motion are

$$\dot{\alpha}\approx u_1, \qquad \dot{\rho}\approx 0$$

$$\dot{\beta}\approx u_2, \qquad \dot{S}\approx 0 \qquad (4.6)$$

$$\dot{\nu}\approx u_3, \qquad \dot{F}\approx 0$$

where we have used the total Hamiltonian

$$\mathcal{H}_t=\mathcal{H}+\sum_{a_1}^{A} u_{a_1} f_{a_1}^1$$

Therefore, due to equations (4.6), there are no secondary constraints.

Physical Variables. By definition, f_1^1, f_2^1, and f_3^1 are primary first-class constraints:

$$\{\rho, S\} \approx 0, \qquad \{\rho, F\} \approx 0, \qquad \{S, F\} \approx 0 \tag{4.7}$$

and as physical variables, ρ, S, and F are first class.

According to the theorem enunciated in Section 2, for any physical variable g, we have that

$$\{g, \rho\} \approx 0, \qquad \{g, S\} \approx 0, \qquad \{g, F\} \approx 0 \tag{4.8}$$

Thus, we obtain the following result:

Theorem. If a variable g in the two-fluid model of Landau is physical, then

$$\frac{\partial g}{\partial \alpha} \approx 0, \qquad \frac{\partial g}{\partial \beta} \approx 0, \qquad \frac{\partial g}{\partial \nu} \approx 0 \tag{4.9}$$

In particular, the Hamiltonian density cannot depend on these variables. Also, $\nabla \alpha$, $\nabla \beta$, and $\nabla \nu$ are physical variables.

From the above result, we can deduce that the time evolution of the velocity $\mathbf{v}_s$ of the superfluid is determined when one knows its value at any time. Equally, the quantity $\mathbf{p} = S \nabla \alpha + F \nabla \nu$ is a physical variable; in this context, our results are compatible with a theorem of Pokrovsky and Khalatnikov (1976): Assume that at some initial instant $t = t_0$ the quantity curl $\mathbf{p}/S$ is equal to zero in all space. Then it remains equal to zero in all the succeeding instants of time.

This theorem is valid if $\mathbf{p}$ is a physical variable, because in the contrary case it would be affected by arbitrary functions of time.

Finally, we observe that there are no second-class constraints, and hence that the Dirac bracket of two quantities g and h coincides with the Poisson bracket, i.e.,

$$\{g, h\}^* = \{g, h\}$$

This implies directly (Dirac, 1958) that we cannot reduce the number of variables of the phase space[3]; thus, all the variables introduced in the Landau–Khalatnikov theory are relevant in the description of the properties of ^{4}He II.

[3] A contrary case occurs in the gauge theory (Chela–Flores et al., to be published).

5. CONCLUSIONS

After a brief summary of some basic concepts, we show that the Hamiltonian density of the system cannot be written as a function of only the canonical variables. We avoid that difficulty by replacing the constraints in the pre-Hamiltonian h. We obtain, remarkably, that the Hamiltonian density $\mathcal{H}$ is weakly zero.

On the other hand, we determine that α, β, and ν are nonphysical variables. Furthermore, $\nabla\alpha$, $\nabla\beta$, and $\nabla\nu$ are physical variables; in this context we show the agreement between our results and a theorem shown by Pokrovsky and Khalatnikov (1976).

Finally, the Dirac bracket coincides with the Poisson bracket, and hence all canonical variables are necessary to describe the superfluid properties (in this model) of ^{4}He II.

TABLE I. Comparison between Gauge Theory of Superfluidity and Landau Two-Fluid Model

Superfluidity	Landau two-fluid model developed via Dirac by the author of this work	Gauge theory of Chela–Flores developed via Dirac by Chela–Flores et al.
Lagrangian density	$\mathcal{L}=\frac{1}{2}\rho v_s^2+\mathbf{p}\cdot\mathbf{v}_s-\epsilon(\rho,S,\mathbf{p})$	$\mathcal{L}=-\frac{i}{2}(\psi\partial_t\psi^*-\psi^*\partial_t\psi)-\frac{U}{2}\lvert\psi\rvert^4-\frac{mk}{2}(\nabla\times\mathbf{A})^2 -\frac{1}{2m}(\nabla+im\mathbf{A})\psi^*\cdot(\nabla-im\mathbf{A})\psi$
Primary constraints	$f_1\approx\rho\approx0, f_2\approx S\approx0, f_3\approx F\approx0$	$f_1=\Pi-\frac{i}{2}\psi^*\approx0, f_2=\Pi^*+\frac{i}{2}\psi\approx0, \mathbf{f}=\mathbf{\Pi}\approx0$
Secondary constraints	no	$\mathbf{f}_s=\mathbf{A}\lvert\psi\rvert^2+k\nabla\times(\nabla\times\mathbf{A})-\frac{1}{2mi}(\psi^*\nabla\psi-\psi\nabla\psi^*)$
Hamiltonian density	$\mathcal{H}\approx0$	$\mathcal{H}=\frac{1}{2m}\nabla\psi^*\nabla\psi-\frac{U}{2}\lvert\psi\rvert^4+\frac{i}{2}\mathbf{A}\psi^*\nabla\psi-\frac{i}{2}\mathbf{A}\psi\nabla\psi^* -\frac{m}{2}\mathbf{A}^2\lvert\psi\rvert^2-\frac{mk}{2}(\nabla\times\mathbf{A})^2$
Physical variables	ρ, S, F	$\psi,\psi^*,\mathbf{A},\Pi,\Pi^*,\mathbf{\Pi}=(\Pi_1,\Pi_2,\Pi_3)$
Gauge variables	α,β,ν	no
Dirac brackets	$\{F,G\}^*=\{F,G\}$	$\{F,G\}^*=\{F,G\}-\sum_{a,a'}^{8}\int d^3x\{F,f_a\}C_{aa'}\{f_{a'},G\}$
First-class constraints	all	no
Second-class constraints	no	all

I consider it necessary to include, for completeness a comparative scheme (see Table I) between the gauge theory of superfluidity (which we studied on another occasion) and the Landau two-fluid model (which we are studying at present) from the point of view of the Dirac mechanics.

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REFERENCES

Bergmann, P. G., and Goldberg, I. (1955). *Physical Review*, **98**, 531.

Chela-Flores, J. (1975). *Journal of Low Temperature Physics*, **21**, 307.

Chela-Flores, J. (1976). *Journal of Low Temperature Physics*, **23**, 775.

Chela-Flores, J., Kálnay, A., Rodriguez Gómez, J., Rodriguez Núñez, J., and Tascón, R. (to be published) "Gauge Fields, Dirac Mechanics and Low Temperature Physics" and "Gauge Field and the Hydrodynamics of Superfluid Helium Four."

Dirac, P. A. M. (1950). *Canadian Journal of Mathematics*, **2**, 129.

Dirac, P. A. M. (1951). *Canadian Journal of Mathematics*, **3**, 1.

Dirac, P. A. M. (1958). *Proceedings of the Royal Society of London*, **A246**, 326.

Dirac, P. A. M. (1964). *Lectures on Quantum Mechanics*, Belfer Graduate School of Sciences Monograph Series N° 2, Yeshiva University, New York.

Dirac, P. A. M. (1966). *Lectures on Quantum Field Theory*, Belfer Graduate School of Sciences Monograph Series N° 3, Yeshiva University, New York.

Kálnay, A., and Ruggeri, G. J. (1973). *International Journal of Theoretical Physics*, **8**, 189.

Khalatnikov, I. M. (1952). *Zhurnal Eksperimental'noi i Teoreticheskoi Fiziki*, **23**, 169.

Landau, L. D. (1941). *Journal of Physics, USSR*, **5**, 71.

Lhuillier, D., Francois, M., and Karatchentzeff, M. (1975). *Physical Review B*, **12**, 7, 2656.

Pokrovsky, V. L., and Khalatnikov, I. M. (1976). *JETP Letters*, **23**, 599.

Sudarshan, E. C. G., and Mukunda, N. (1974). *Classical Dynamics, a Modern Perspective*. John Wiley & Sons, New York.